Deep Learning

3.4 Backpropagation

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Recap

1. Gradient of a scalar valued function $f(\mathbf{x})$: $\mathbf{x} \rightarrow \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_D}\right)$

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2. Gradient of a vector valued function $\mathbf{f}(\mathbf{x})$ is called Jacobian:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_m \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

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2. For applying Gradient descent, we need gradient of individual sample loss with respect to all the model parameters

$$l_n = l(f(x_n; W, \mathbf{b}), y_n)$$

$\frac{\partial l_n}{\partial W_{i,j}^{(l)}}$ and $\frac{\partial l_n}{\partial \mathbf{b}_i^{(l)}}$

Forward pass operation

$$x^{(0)} = x \xrightarrow{W^{(1)},\mathbf{b}^{(1)}} s^{(1)} \xrightarrow{\sigma} x^{(1)} \xrightarrow{W^{(2)},\mathbf{b}^{(2)}} s^{(2)} \dots x^{(L-1)} \xrightarrow{W^{(L)},\mathbf{b}^{(L)}} s^{(L)} \xrightarrow{\sigma} x^{(L)} = f(x;W,\mathbf{b})$$

Formally, $x^{(0)} = x, f(x;W,\mathbf{b}) = x^{(L)}$

$$\forall l = 1,\dots,L \quad \begin{cases} s^{(l)} & = W^{(l)}x^{(l-1)} + \mathbf{b}^{(l)} \\ x^{(l)} & = \sigma(s^{(l)}) \end{cases}$$

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4. $$J_{f_N \circ f_{N-1} \circ \ldots f_1}(x) = J_{f_N}(f_{N-1}(\ldots(x)))\ldots J_{f_3}(f_2(f_1(x))) \cdot J_{f_2}(f_1(x)) \cdot J_{f_1}(x)$$

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- $J_{f(x)}$ is Jacobian of f computed at x.

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$$\frac{\partial \ell}{\partial s_i^{(l)}} = \frac{\partial \ell}{\partial x_i^{(l)}} \frac{\partial x_i^{(l)}}{\partial s_i^{(l)}} = \frac{\partial \ell}{\partial x_i^{(l)}} \sigma'(s_i^{(l)})$$

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$$\frac{\partial \ell}{\partial x_j^{(l-1)}} = \Sigma_i \frac{\partial \ell}{\partial s_i^{(l)}} \frac{\partial s_i^{(l)}}{\partial x_j^{(l-1)}} = \Sigma_i \frac{\partial \ell}{\partial s_i^{(l)}} W_{i,j}^{(l)}$$

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4. $$\frac{\partial \ell}{\partial b_i^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}} \frac{\partial s_i^{(l)}}{\partial b_i^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}} \tag{2}$$

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3. Then wrt the parameters
$$\frac{\partial \ell}{\partial w_{i,j}^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}} x_j^{(l-1)} \text{ and } \frac{\partial \ell}{\partial b_i^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}}$$

Jocobian in Tensorial form

2. $\psi : \mathcal{R}^N \rightarrow \mathcal{R}^M$ then $\left[\frac{\partial \psi}{\partial x}\right] = \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_M}{\partial x_1} & \cdots & \frac{\partial \psi_M}{\partial x_N} \end{bmatrix}$

Jocobian in Tensorial form

1. $\psi : \mathcal{R}^{N\times M} \rightarrow \mathcal{R}$ then $\left[\left[\frac{\partial\psi}{\partial x}\right]\right] = \begin{bmatrix} \frac{\partial\psi}{\partial w_{1,1}} & \cdots & \frac{\partial\psi}{\partial w_{1,M}} \\ \vdots & \ddots & \vdots \\ \frac{\partial\psi}{\partial w_{N,1}} & \cdots & \frac{\partial\psi}{\partial w_{N,M}} \end{bmatrix}$
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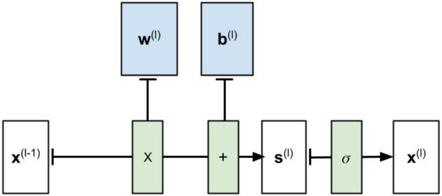
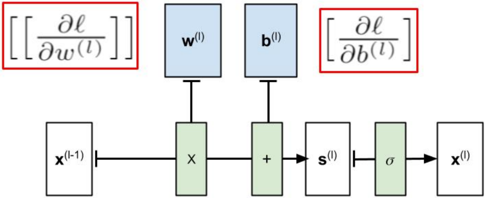
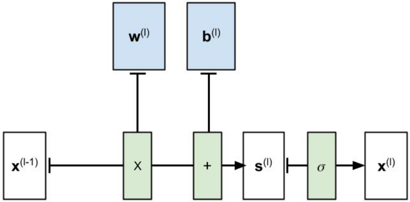
Forward Pass



Goal of Backward Pass



Begin from succeeding layer



$$\boxed{\frac{\partial \ell}{\partial x^{(l)}}}$$

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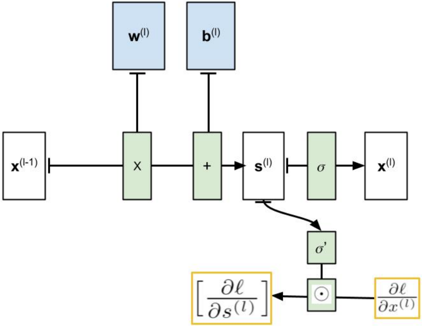

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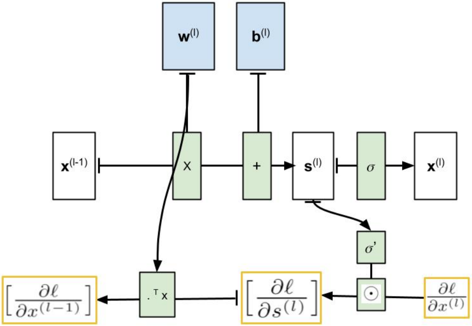

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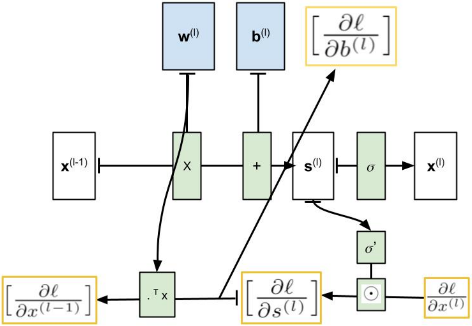

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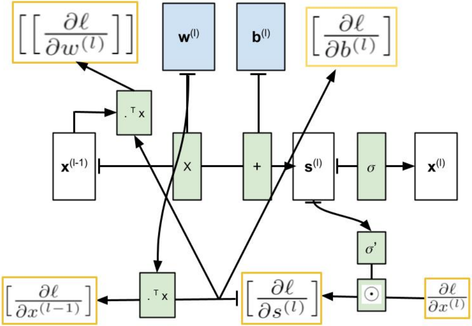

Update the parameters

1. $W^{(l)} = W^{(l)} - \eta \left[\left[\frac{\partial \ell}{\partial w^{(l)}}\right]\right]$ and $\mathbf{b}^{(l)} = \mathbf{b}^{(l)} - \eta \left[\frac{\partial \ell}{\partial b^{(l)}}\right]$

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4. Nonlinearities go into simple element wise operations
5. In an untreated situation, BP Needs all the intermediate layer results to be in memory
6. Takes twice the computations of forward pass